Fernando Soares

4. Binary Mathematical Morphology

Elementary set theory. Properties of sets. Properties of morphological transformations. Notion of structuring element. Neighborhood Transformation. Elementary morphological transformations. Hit-or-miss transformations. Thinning and Thickening. Geodetic transformations. Euler Number (connectivity).

Introduction

Origins:

- Pioneering work by Georges Matheron and Jean Serra (1960s, France).

  Creation of the Center de Morphologie Mathématique de Fontainebleau (1968).

Objective:

- Mathematical morphology is based on set theory and is intended to quantify structures from the geometric point of view.

Introduction

Metodology:

1. Uses the notion of set to represent structures.

2. Transformation of sets to make them measurable:

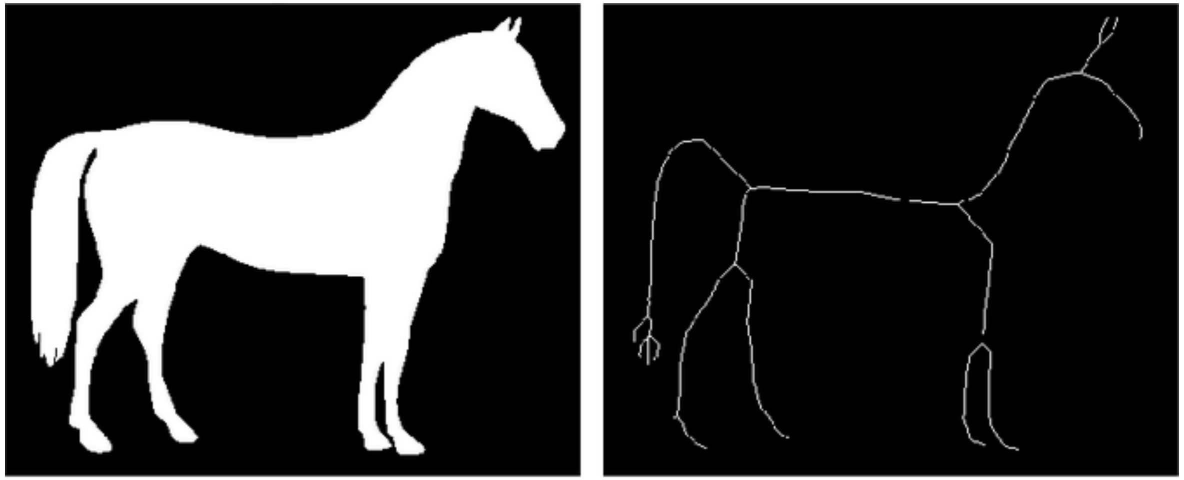
   - Interaction of the set of objects under study with another object with known shape (structuring element).

   - The transformation of the initial set, through successive operations, shows its structural characteristics, which are recorded along the new generated sets, implying that <u>the transformed set is simpler than the original set</u>.

Introduction

Metodology:

3. Making measurements on transformed sets.



Measure:

- Surface
- Shape
- Perimeter
- Etc.

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Introduction

Morphological image processing can be applied in the following contexts:

- Binary morphology: Images are binary (most frequent).
- Numerical morphology: Images can be either gray (monochrome) or color (polychrome).
- The associated terms "Morphology" and "Mathematics" (which constitute the designation of this theory) refer to the use of concepts of set logic and numerical operations.

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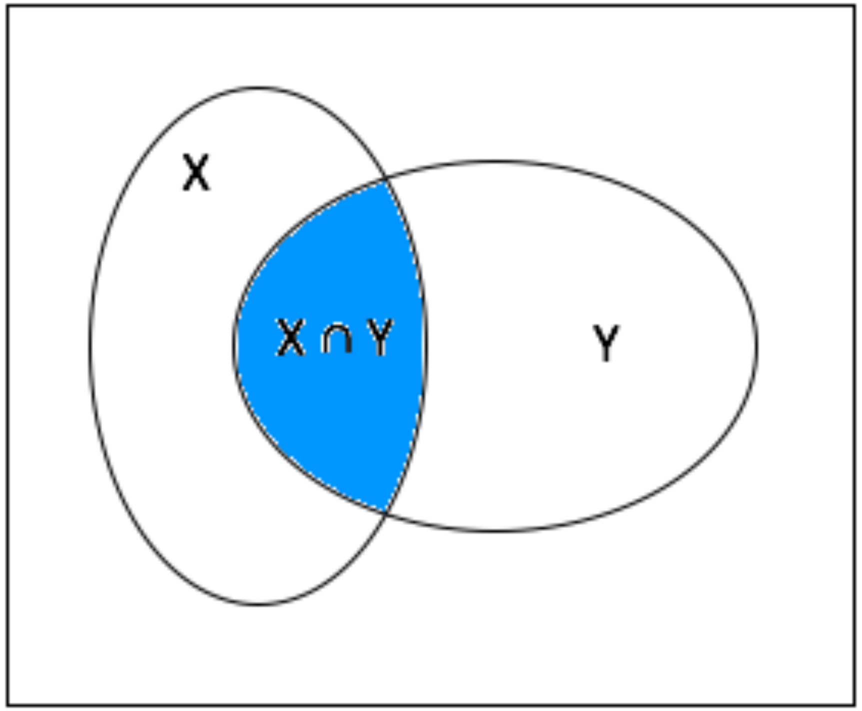
Elementary set theory

Intersection

$$\boxed{X \cap Y = \{x\colon x \in X \wedge x \in Y\}}$$

- Comutative: $X \cap Y = Y \cap X$
- Associative: $X \cap (Y \cap Z) = (Y \cap X) \cap Z$
- Idempotent: $X \cap X = X$



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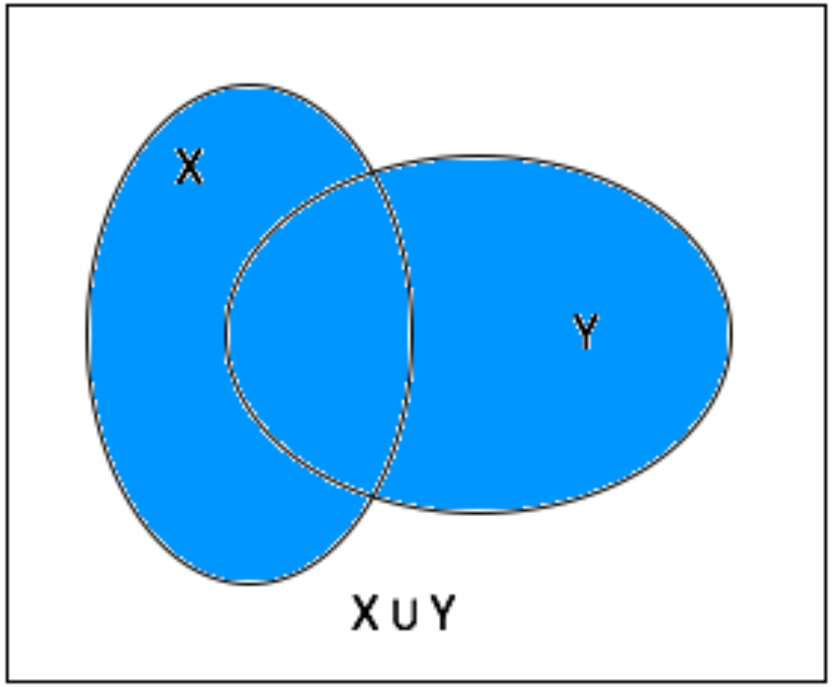
Elementary set theory

Reunion

$$X \cup Y = \{x : x \in X \vee x \in Y\}$$

- Comutative: $X \cup Y = Y \cup X$
- Associative: $X \cup (Y \cup Z) = (Y \cup X) \cup Z$
- Idempotent: $X \cup X = X$



Elementary set theory

Relationship between intersection and reunion (distributivity):

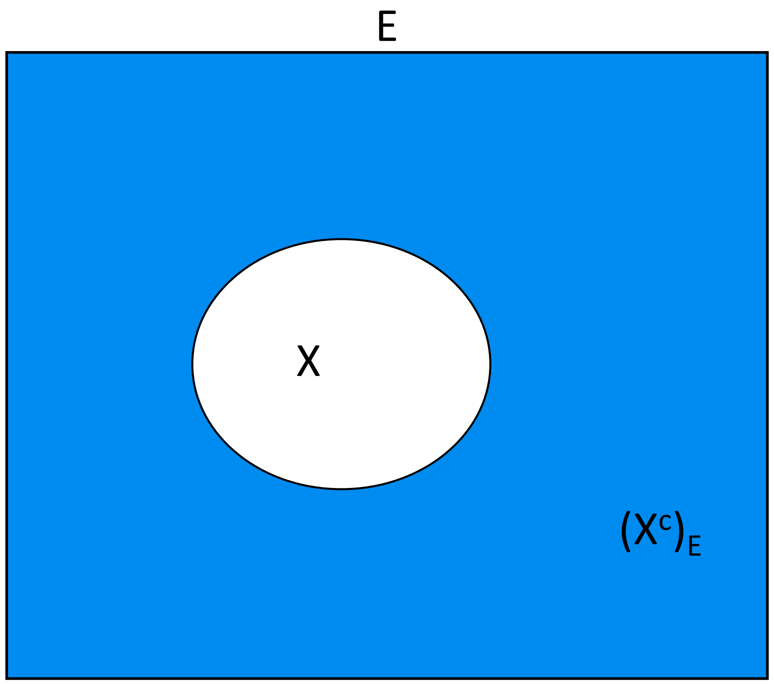
$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

Elementary set theory

Complementary Set

$$(X^c)_E = \{x : x \in E \wedge x \notin X\}$$



- Morgan formulas:

$$\left((X \cap Y)^c\right)_E = (X^c)_E \cup (Y^c)_E$$

$$\left((X \cup Y)^c\right)_E = (X^c)_E \cap (Y^c)_E$$

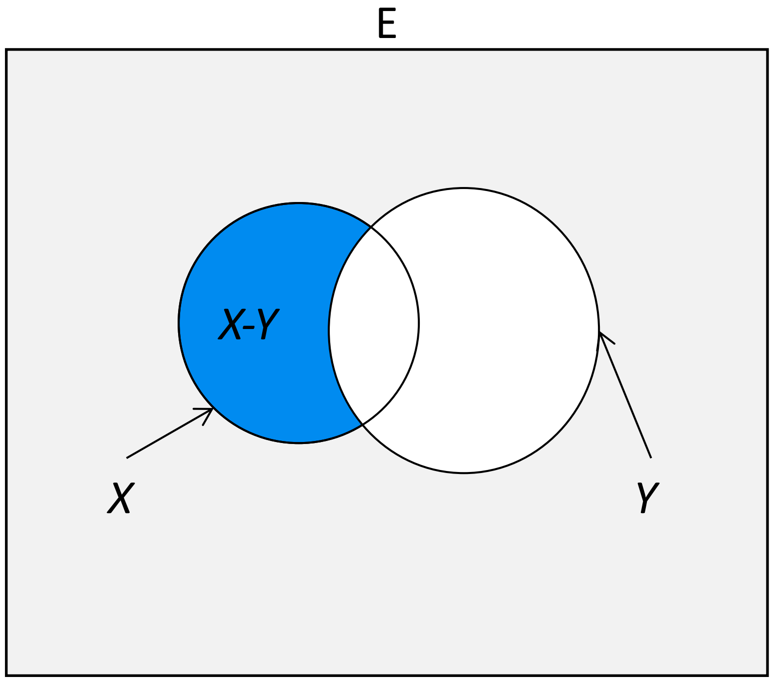
Elementary set theory

Logical difference

$$X - Y = \{x : x \in X \wedge x \notin Y\}$$

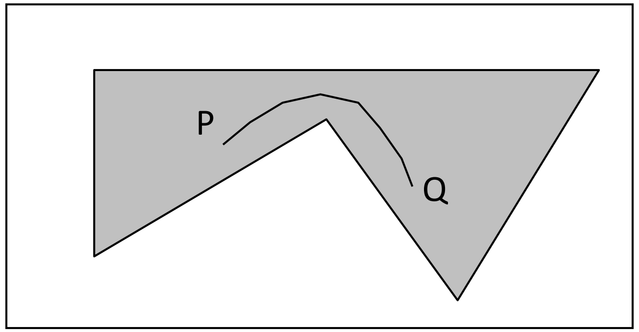
or

$$X - Y = X \cap (X \cap Y)^c$$

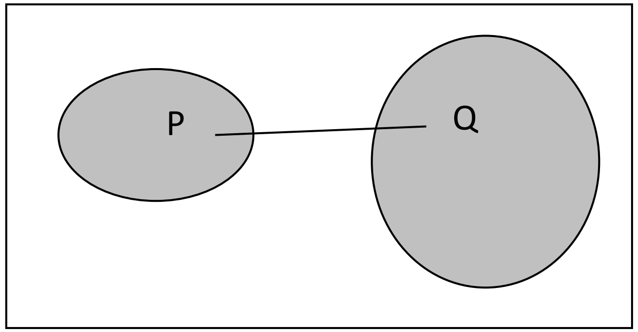


Set Properties

Connectivity: a set X is said to be connected if, for any two points $P(x_i, y_i)$ and $Q(x_m, y_n)$ included in it, there is at least one path that joins them and which is fully included in X.



X Connected



X not connected

The connectivity of an array depends, however, on how the connection between pixels in a digital grid is defined (in square digital grids with 4-connectivity and 8-connectivity, or hexagonal with 6-connectivity).

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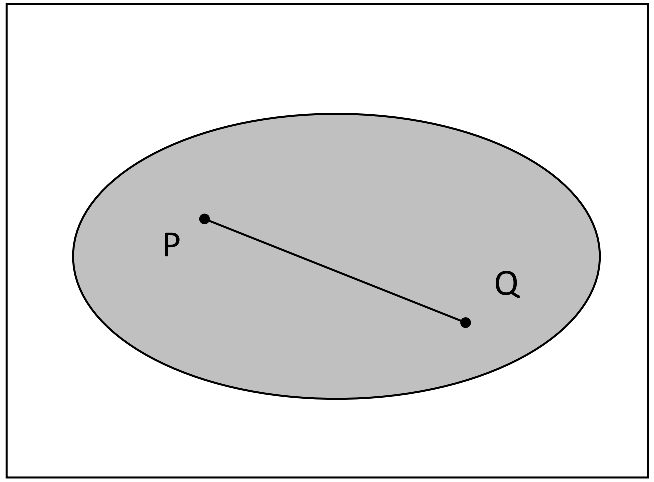
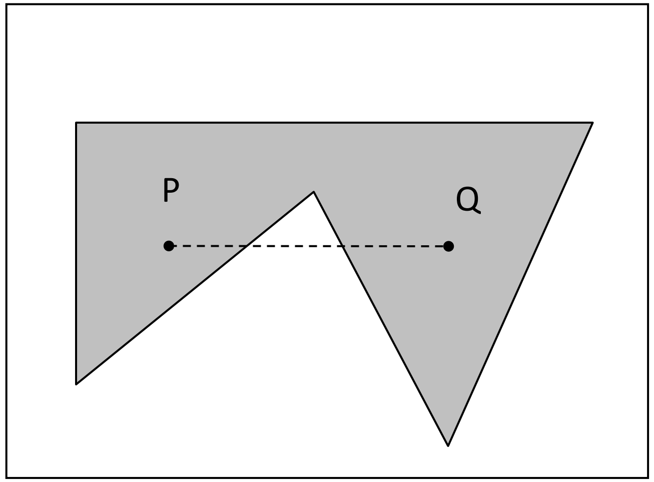
Set Properties

Convexity: a set X is said to be convex if, whatever pair of points $P(x_i, y_i)$ and $Q(x_m, y_n)$ are included in it, the line segment joining them is fully included in X.



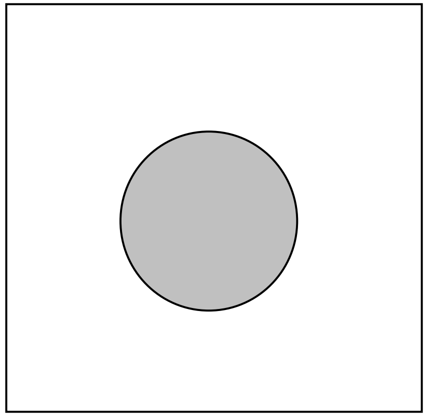
X convex



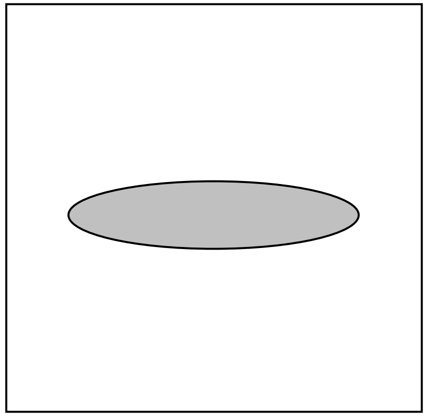
X not convex

Set Properties

Isotropy: A set X is said to be isotropic if it is evenly spread in all directions.



X isotropic



X anisotropic

Predominating a given direction, it is an anisotropic set. This concept is associated with the spatial structure of objects.

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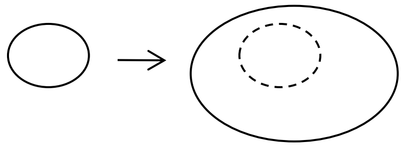
Properties of morphological transformations

If φ is a morphological transformation then obeys one or more of the following properties :

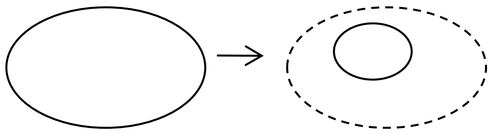
- φ is **extensive** if the resulting set contains initial set X.

$$\boxed{X \subset \varphi(X)}$$



- φ is **anti-extensive** if the resulting set is contained in the initial set X.

$$\boxed{\varphi(X) \subset X}$$



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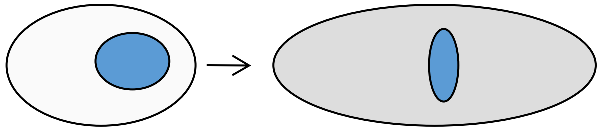
Properties of morphological transformations

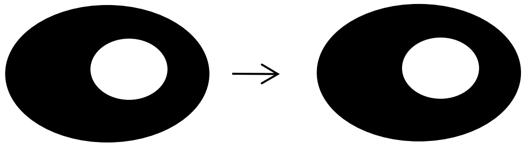
- φ is **increasing** if the inclusion relationship between the initial and resulting sets is maintained.

$$\boxed{Y \subset X \Rightarrow \varphi(Y) \subset \varphi(X)}$$



- φ is **idempotent** if its successive application to X does not change it.

$$\boxed{\varphi(\varphi(X)) \equiv X}$$



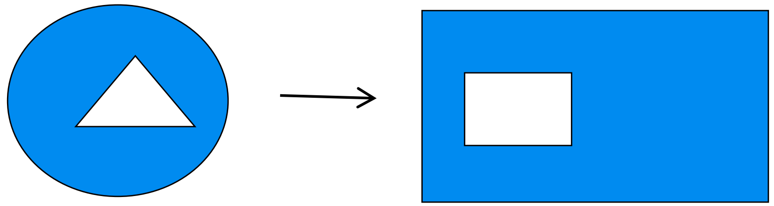
Properties of morphological transformations

- Two morphological transformations φ_1 and φ_2 are **dual** if:

$$\boxed{\varphi_1(X) \equiv \varphi_2(X^c)}$$

- Finally, a transformation is said to be **homotopic** if it does not modify the connectivity number E of a set X, that is:

$$\boxed{E[\varphi(X)] = E[X]}$$



Notion of structuring element

Structuring element (B): This is a particular case of binary image, being usually small and simple (eg $B_{3\times3}$ with all values equal to one).

A morphological transformation can only be accomplished with the prior definition of a structuring element.

As its name implies, when associated with a certain morphological transformation, B runs through the image in order to assess whether or not it "fits" the objects present in it. In the process, you can modify the shape and topological characteristics of these objects (for example, the connectivity or convexity).

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Notion of structuring element

The shape of B can be any, the most common being the square, disk, segment, circle, pair of dots and hexagon (in the hexagonal digital grid), which are chosen according to the desired objectives. For example:

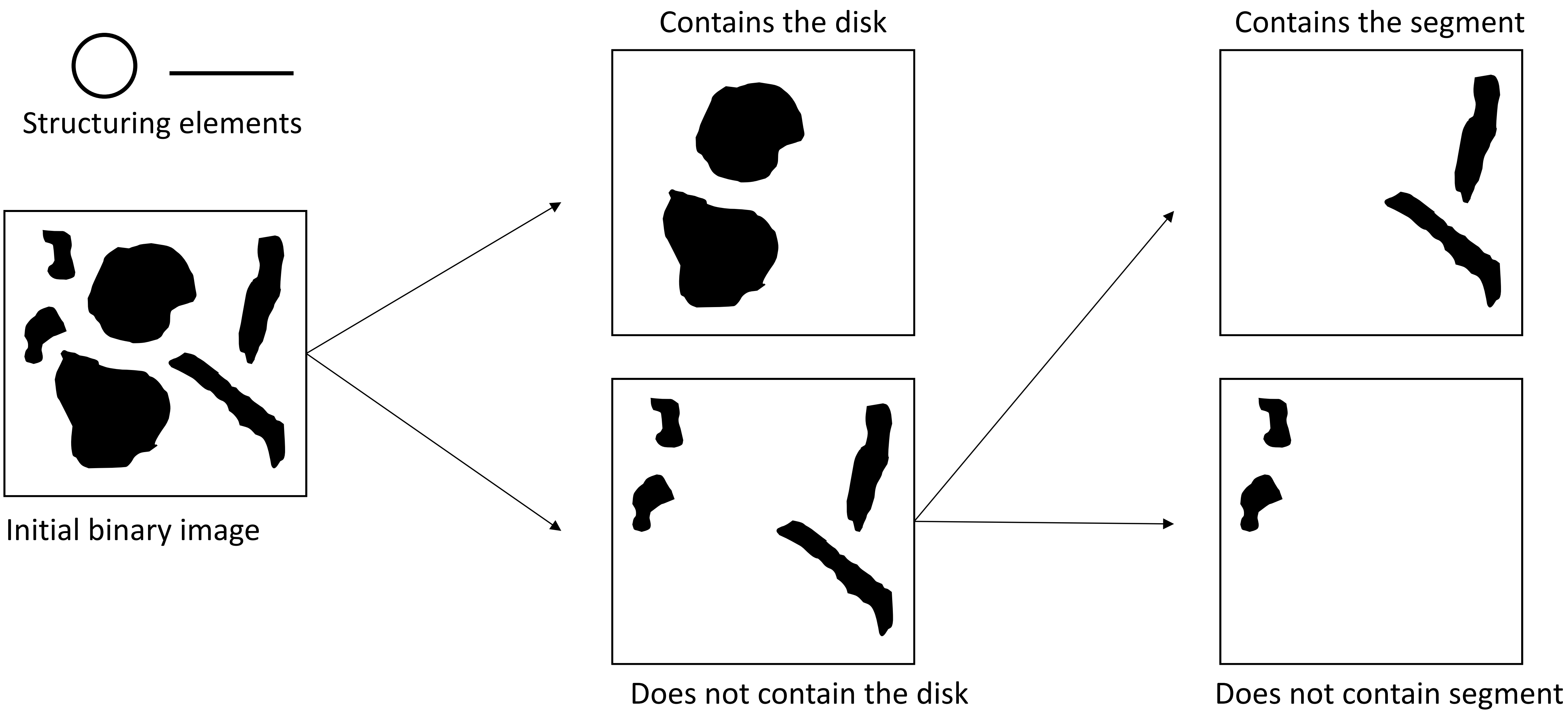
DISC: determination of the size distribution of objects (particle size).

SEGMENT: detection of preferential alignments.

PAIR OF POINTS: characterization of the dispersion state (covariance).

CIRCUMFERENCE: study of the neighborhood of a point (neighborhood transformation).

Notion of structuring element



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Notion of structuring element

The center, or reference pixel of B is generally its geometric center, however any other point can be defined for this purpose.

The center of B marks its position over the initial image and therefore the position of the transformed pixel.

The distribution of pixel values within B is called the neighborhood transformation (V).

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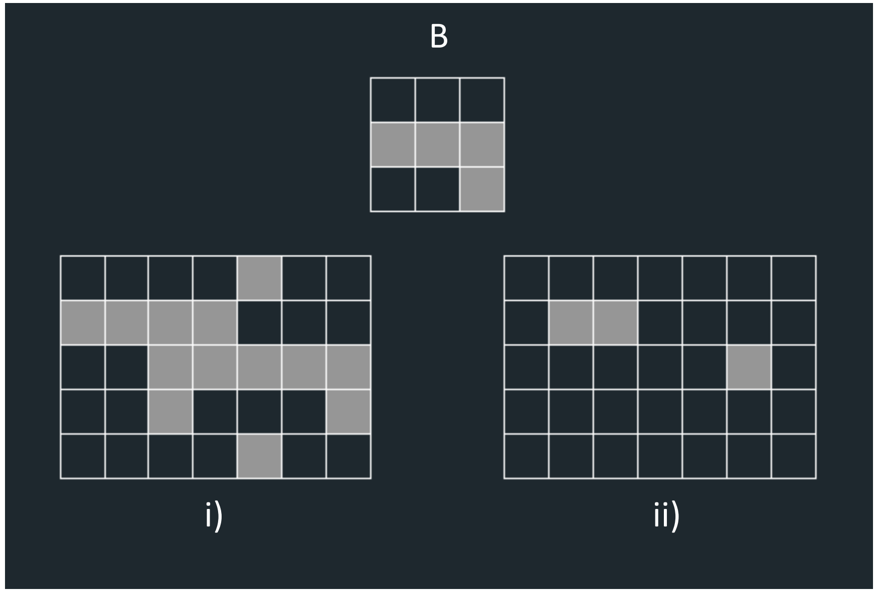
Neighborhood Transformation

A **neighborhood transformation** consists of identifying / altering a pixel of an image if a given neighborhood setting V is found around that pixel.

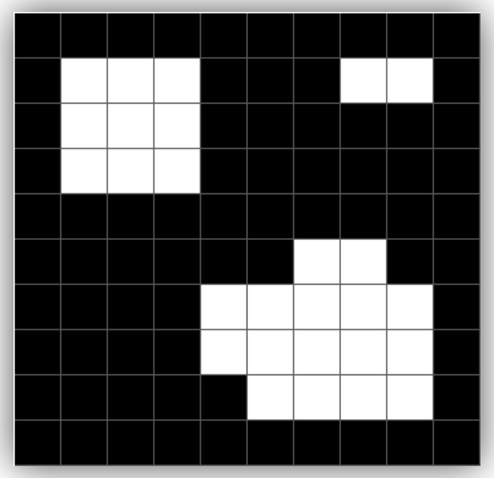
- In the figure, there is an elementary square B that will traverse all the pixels of the image i).
- In ii) are marked the positions in which the neighborhood configuration V (B_x) is identical to that of B.
- The most common neighborhood transformations are thinning and thickening.



Elementary morphological transformations

The morphological **erosion** transformation (ε) of a given set X by a structuring element with its origin in x (B_x) is defined by the following expression:

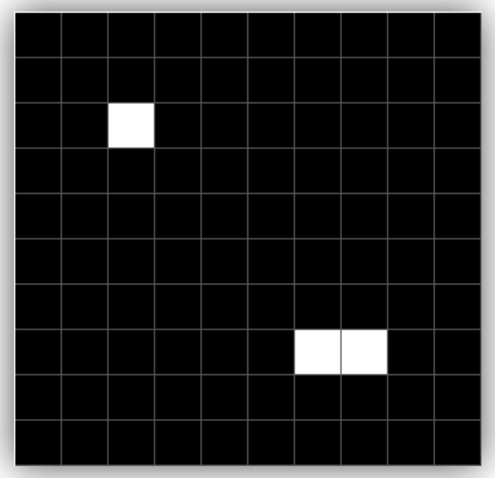
$$\varepsilon_B(X) = \{x : B_x \subset X\}$$



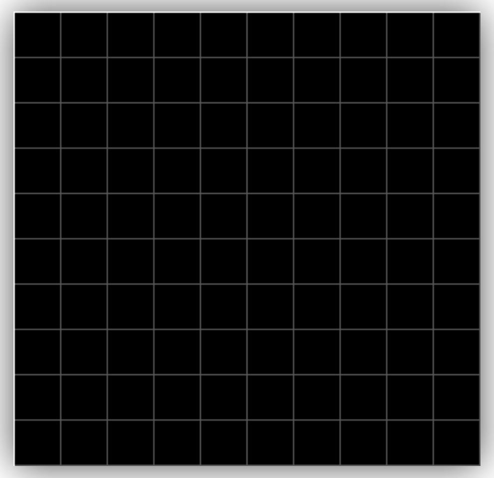
X

B

1	1	1
1	1	1
1	1	1



$\varepsilon_{B_{x(3\times3)}}(X)$

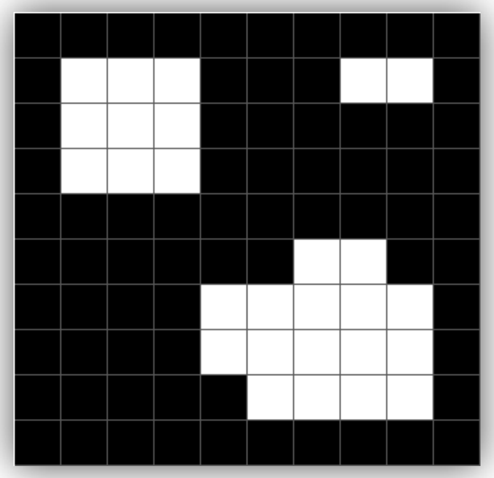


$\varepsilon_{B_{x(5\times5)}}(X)$

Elementary morphological transformations

The morphological **dilation** transformation (δ) of a given set X by a structuring element with its origin in x (B_x) is defined by the following expression:

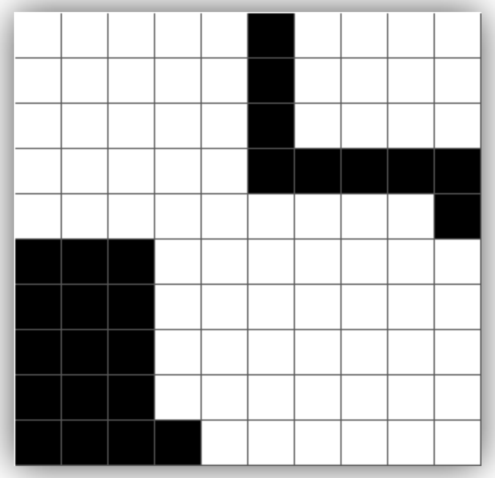
$$\delta_B(X) = \{x: B_x \cap X \neq \phi\}$$



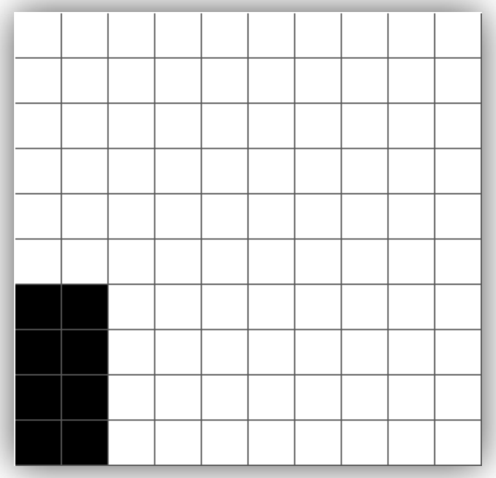
X

B

1	1	1
1	1	1
1	1	1



$\delta_{B_{x(3\times3)}}(X)$

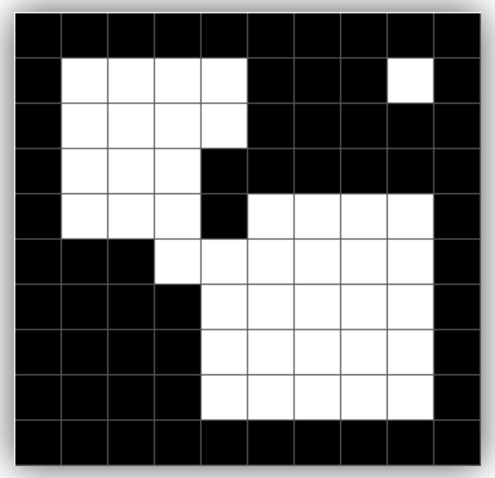


$\delta_{B_{x(5\times5)}}(X)$

Elementary morphological transformations

The **opening** (γ) of X consists in performing the dilation of the erosion result of the set X.

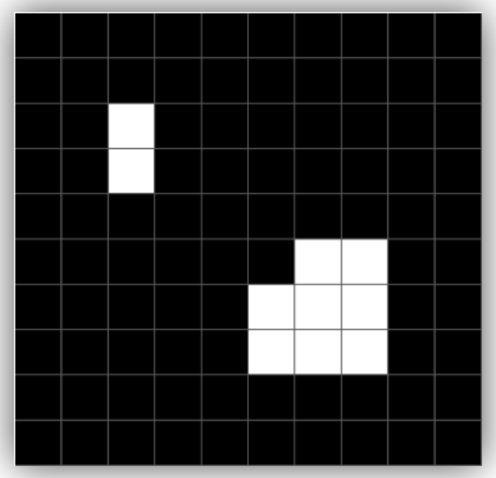
$$\gamma_B(X) = \delta_B(\varepsilon_B(X))$$



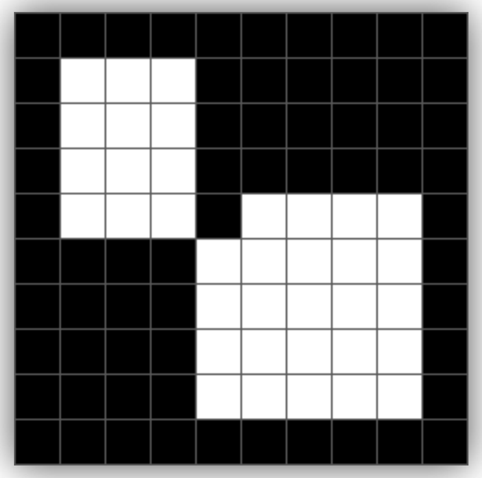
X

B

1	1	1
1	1	1
1	1	1



$\varepsilon_{B_{x(3\times3)}}(X) = Y$

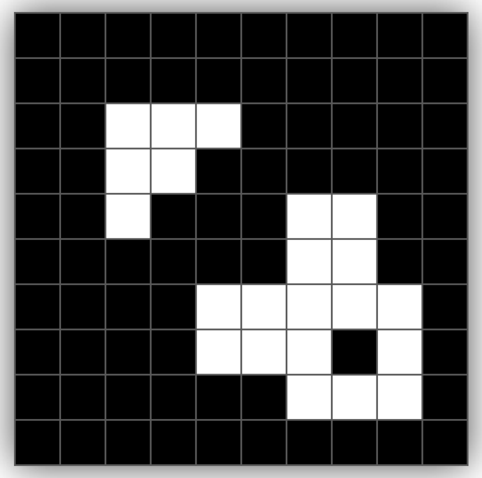


$\delta_{B_{x(3\times3)}}(Y) = \gamma_B(X)$

Elementary morphological transformations

The **closing** (ϕ) of X consists of erosion of the result of dilation of set X.

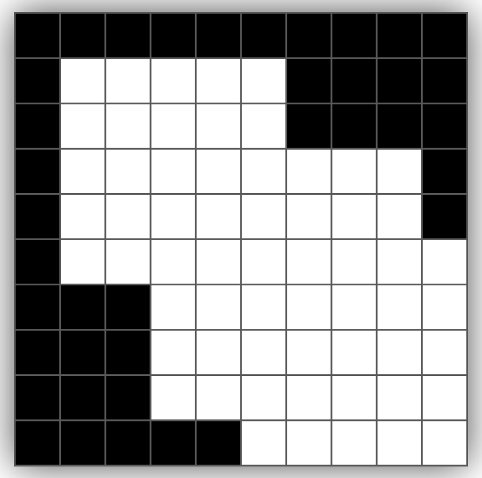
$$\phi_B(X) = \varepsilon_B(\delta_B(X))$$



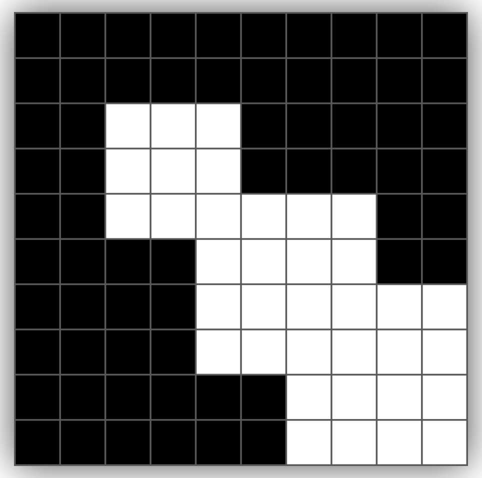
X

B

1	1	1
1	1	1
1	1	1



$\delta_{B_{x(3\times3)}}(X) = Y$



$\varepsilon_{B_{x(3\times3)}}(Y) = \phi_B\ (X)$

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Elementary morphological transformations

- Properties table of elementary morphological transformations:

	Extensiva	Anti-extensiva	Crescente	Idempotente	Homotópica	Transformação dual
Erosão	-	x	x	-	-	*Dilatação*
Dilatação	x	-	x	-	-	*Erosão*
Abertura	-	x	x	x	-	*Fecho*
Fecho	x	-	x	x	-	*Abertura*

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Hit-or-Miss transformations

The Hit-or-Miss transformation (HMT) applied to X consists of a neighborhood transformation using a composite structuring element B = (B_1, B2), with B1 ∩ B2 = ϕ, and which results from the simultaneous verification of the following conditions: <u>B_1 matches X and B_2 matches X^c</u>.

$$\boxed{HMT_B(X) = \{x: (B_1)_x \subseteq X \wedge (B_2)_x \subseteq X^c\}}$$

$$\boxed{HMT_B(X) = \varepsilon_{B_1}(X) \cap \varepsilon_{B_2}(X^c)}$$

- The indices of the composite structuring element are generally three: “1” (∈ B_1 domain), “-1” (∈ B_2 domain) and “0” (indifferent).

- HMT is generally used to find specific configurations in groups of pixels or objects and is determined by the intersection between erosions of X by B_1 and X^c by B_2 (as will be seen in some examples below).

Hit-or-Miss transformations (examples)

<u>Isolated points</u> (pixels without any other pixels in its neighborhood).

Two configurations are drawn (SE) B_1 and B_2, such as, $B_1 \subseteq X$ e $B_2 \subseteq X^c$.

B1

0	0	0
0	1	0
0	0	0

B2

1	1	1
1	0	1
1	1	1

Opting for a composite B has the following configuration:

B

-1	-1	-1
-1	1	-1
-1	-1	-1

1 = belongs
-1 = belongs to the complementar
0 = indifferent

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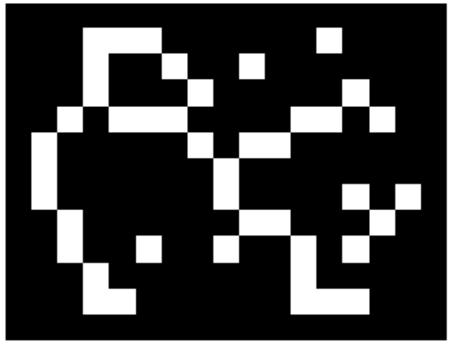
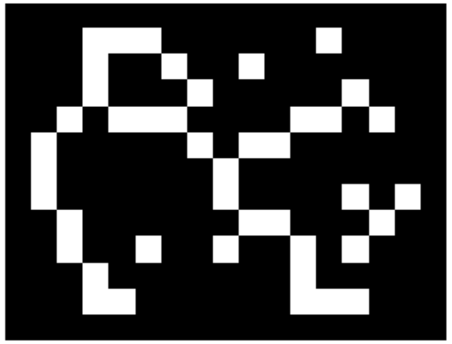
Hit-or-Miss transformations (examples)

Inicial

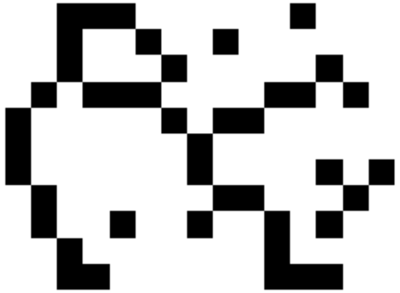


E1

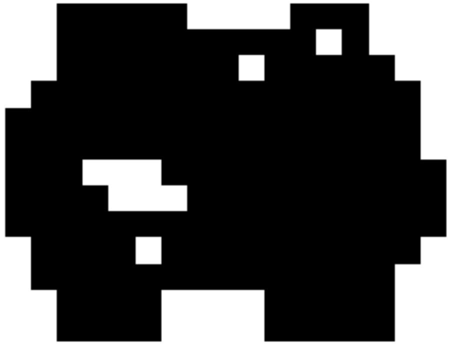


$\varepsilon_{B_1}(X) = \{x : B_1 \subset X\}$

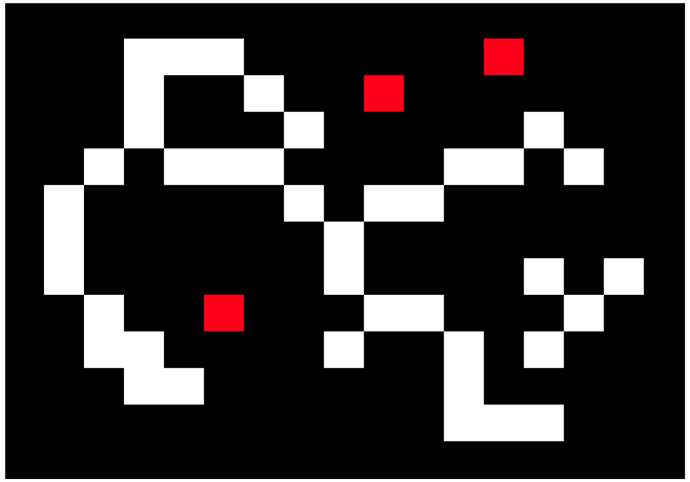
Complementar



E2



$\varepsilon_{B_2}(X^c) = \{x : B_2 \subset X^c\}$



$Isolated\ points = \varepsilon_{B_1} \cap \varepsilon_{B_2}$

Hit-or-Miss transformations (examples)

End points (pixels with a maximum of one pixel in its neighborhood).

Using B_1 e B_2:

B1(90º)

-1	1	-1
-1	1	-1
-1	-1	-1

B2(90º)

-1	-1	-1
1	-1	1
1	1	1

B1(45º)

1	-1	-1
-1	1	-1
-1	-1	-1

B2(45º)

-1	1	1
1	-1	1
1	1	1

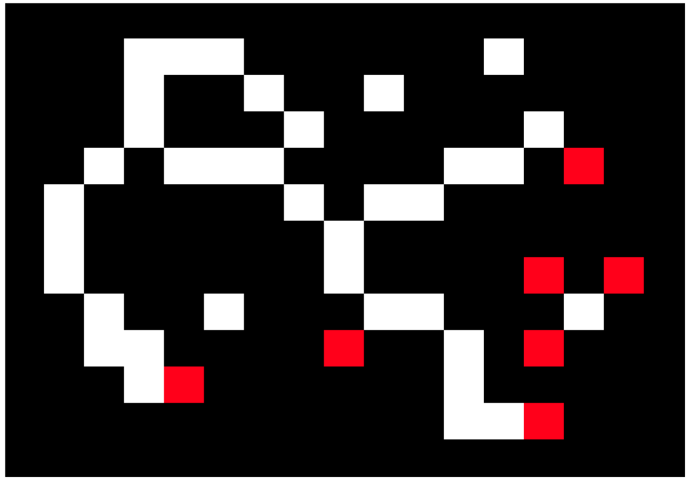
Using B composite:

B(90º)

0	1	0
-1	1	-1
-1	-1	-1

B(45º)

1	-1	-1
-1	1	-1
-1	-1	-1



End points

Hit-or-Miss transformations (examples)

Multiple points (pixels with more than two pixels in its neighborhood).

Using B_1 e B_2:

B1(90º)

-1	1	-1
-1	1	-1
1	-1	1

B2(90º)

-1	-1	-1
1	-1	1
-1	1	-1

B1(45º)

1	-1	1
-1	1	-1
1	-1	-1

B2(45º)

-1	-1	-1
-1	-1	1
-1	1	1

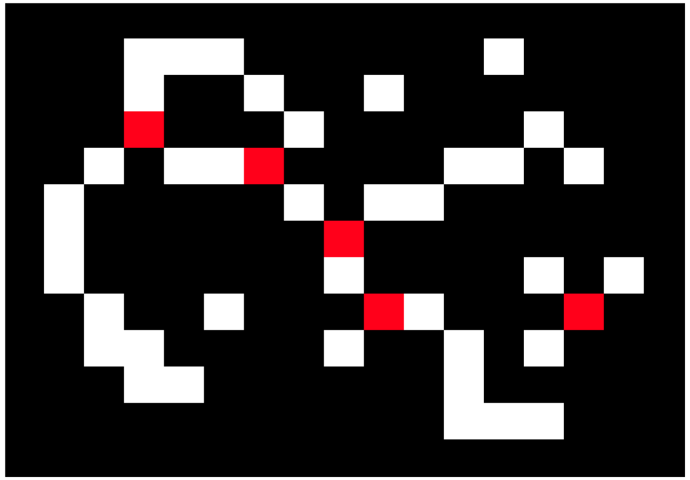
- Using B composite:

B(90º)

0	1	0
-1	1	-1
1	-1	1

B(45º)

1	0	1
0	1	-1
1	-1	-1



Multiple points

Hit-or-Miss transformations (examples)

Right corners (pixels that form a convex right angle).

Using B_1 e B_2:

B1

-1	1	-1
-1	1	1
-1	-1	-1

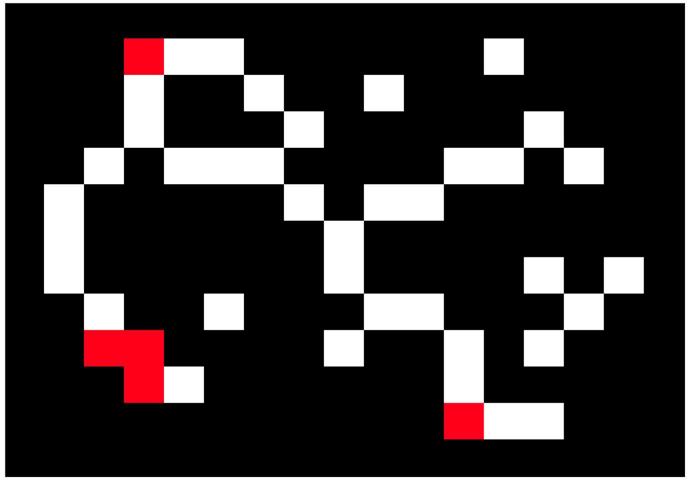
B2

-1	-1	-1
1	-1	-1
1	1	-1

Using B composite:

B

0	1	0
-1	1	1
-1	-1	0



Right corners

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Hit-or-Miss transformations (examples)

<u>Contours</u> (pixels with at least one pixel belonging to the complementary set).

Using B_1 e B_2:

B1(90º)

-1	-1	-1
-1	1	-1
-1	-1	-1

B2(90º)

-1	1	-1
-1	-1	-1
-1	-1	-1

B1(45º)

-1	1	1
1	1	1
1	1	1

B2(45º)

1	-1	-1
-1	-1	-1
-1	-1	-1

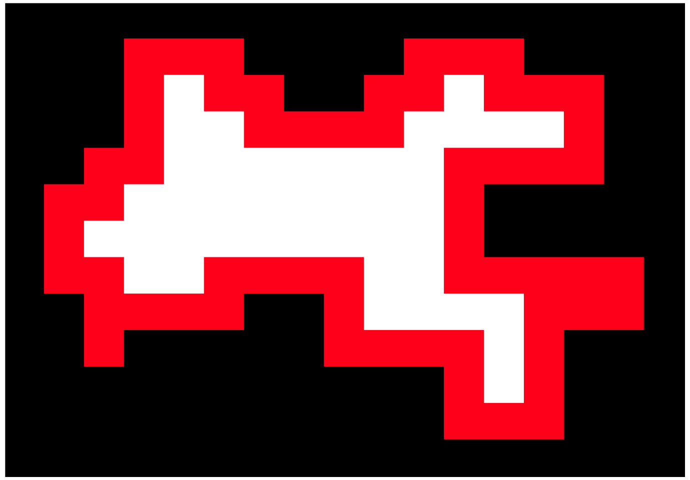
Using B composite:

B(90º)

0	-1	0
0	1	0
0	0	0

B(45º)

-1	0	0
0	1	0
0	0	0



$Contour\ points$

Thinning Transformation

Thinning (THIN) of an X set: consists of a neighborhood transformation that removes from X all points that correspond to a given neighborhood configuration $V(B_x)$.

$$\boxed{THIN(X,B) = X \cap NOT[HMT(X,B)]}$$

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Thinning Transformation

Aims to remove region / object pixels from the image.

It is applied to binary images only and produces a binary image as a result.

In general, the thinning operation is determined by translating the origin of the structuring element B by all the pixels of the image, comparing each one of its neighborhood configuration with the configuration of the corresponding pixels in the image.

Thinning Transformation

If both settings match, then the image pixel corresponding to the center position of B is assigned the value 0; otherwise it remains unchanged.

Morphological erosion and opening are examples of thinning transformations.

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Thinning Transformation

"**Skeletonization**" is an example of a morphological thinning process that aims to reduce the regions of a binary image to a minimal structure that preserves the extent and connectivity of the original regions.

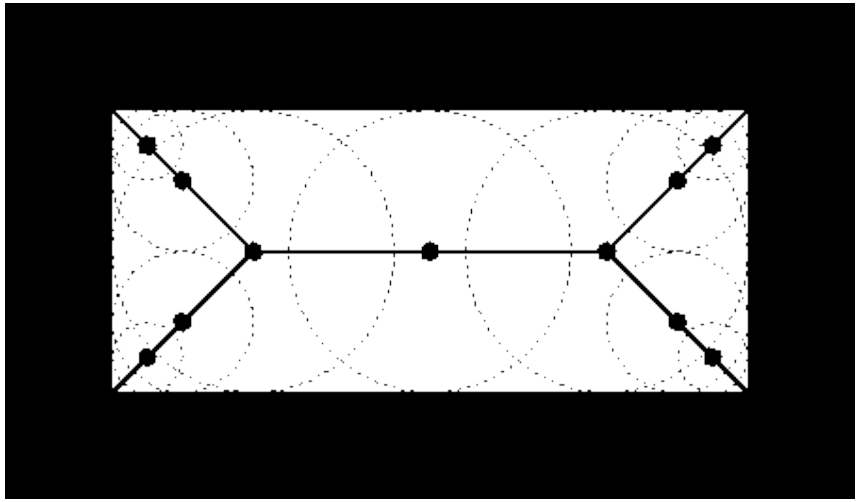
- It is a transformation often used to “narrow” boundary detection results by reducing the thickness of lines to others that are only one pixel thick.

  Note that the resulting skeleton is a connected set is also a connected set.

Thinning Transformation

The skeleton of an assembly can be determined in several ways:

a) Location of the centers of maximum circumferences bi- tangent to the limits of the considered region.

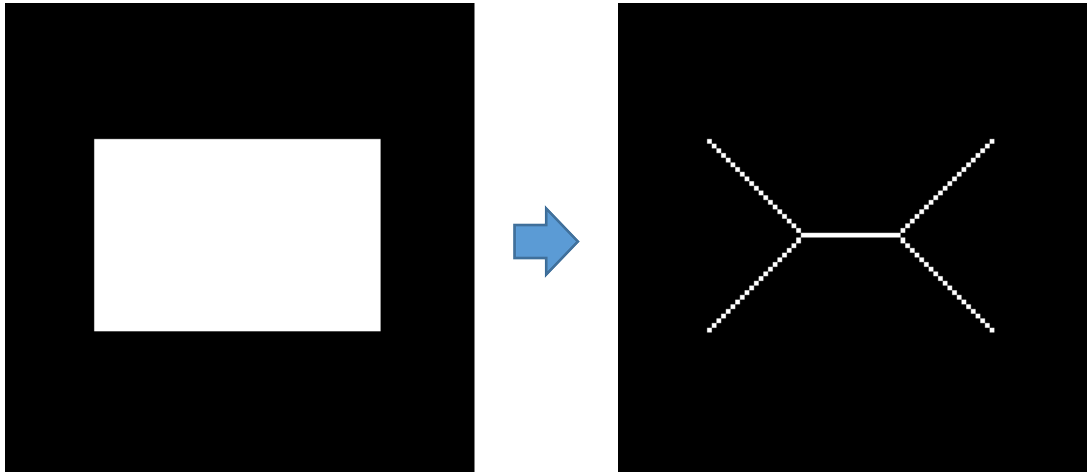


Thinning Transformation

b) Lantuéjoul formula: For a discrete binary image X, the S(X) skeleton is the union of all subsets S_k (X) with a structuring element B of dimension k.

$$S(X) = \bigcup_{k}^{K} S_k(X)$$

$$S_k(X) = \varepsilon_{kB}(X) \cap NOT[\gamma_B(\varepsilon_{kB}(X))]$$



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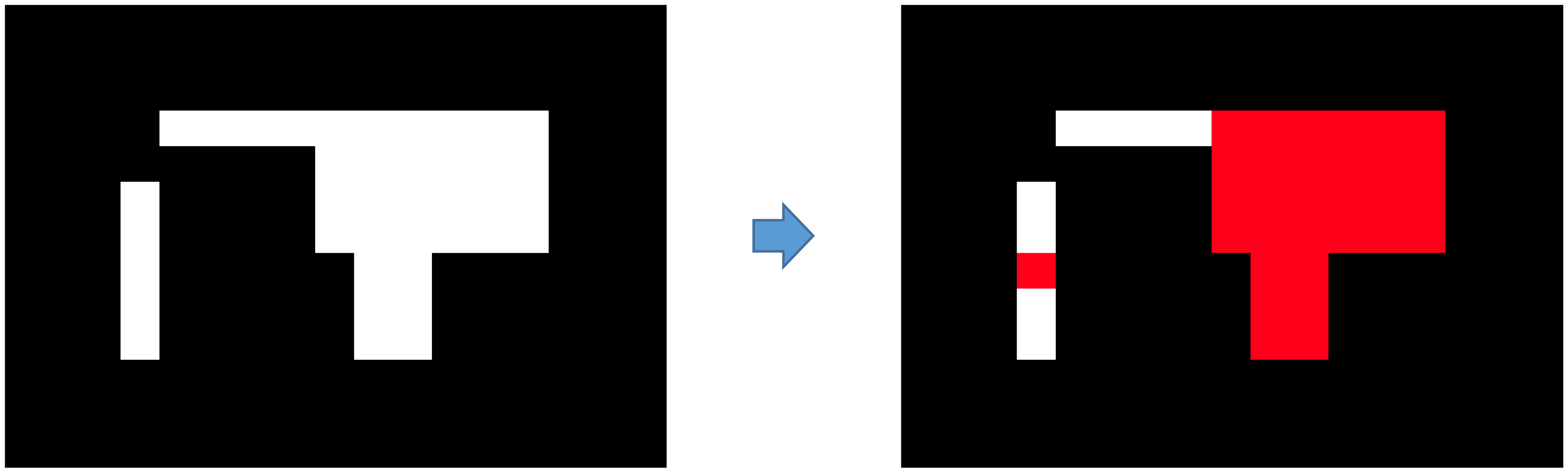
Thinning Transformation

Prune is also a thinning operation that aims to successively suppress the extreme points of a binary set until the idempotence condition is met.



Thickening Transformation

Thickenening (THICK) of a set X: This is a neighborhood transformation that adds to X all points that correspond to a given neighborhood setting V (Bx).

$$\boxed{THICK(X,B) = X \cup HMT(X,B)}$$

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Thickening Transformation

Aims to grow regions / objects by adding pixels to them.

It is applied to binary images only and produces a binary image as a result.

It is used in many applications, including the determination of the convex space of a certain set of points (convex hull), or the determination of the skeleton by influence zone (SKIZ).

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Thickening Transformation

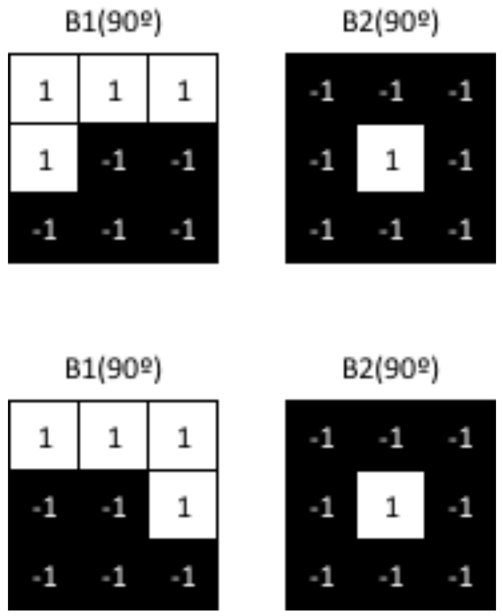
In general, the thickening operation is determined by translating the origin of the structuring element B by all the pixels of the image, comparing each one of its neighborhood configuration with the configuration of the corresponding pixels in the image.

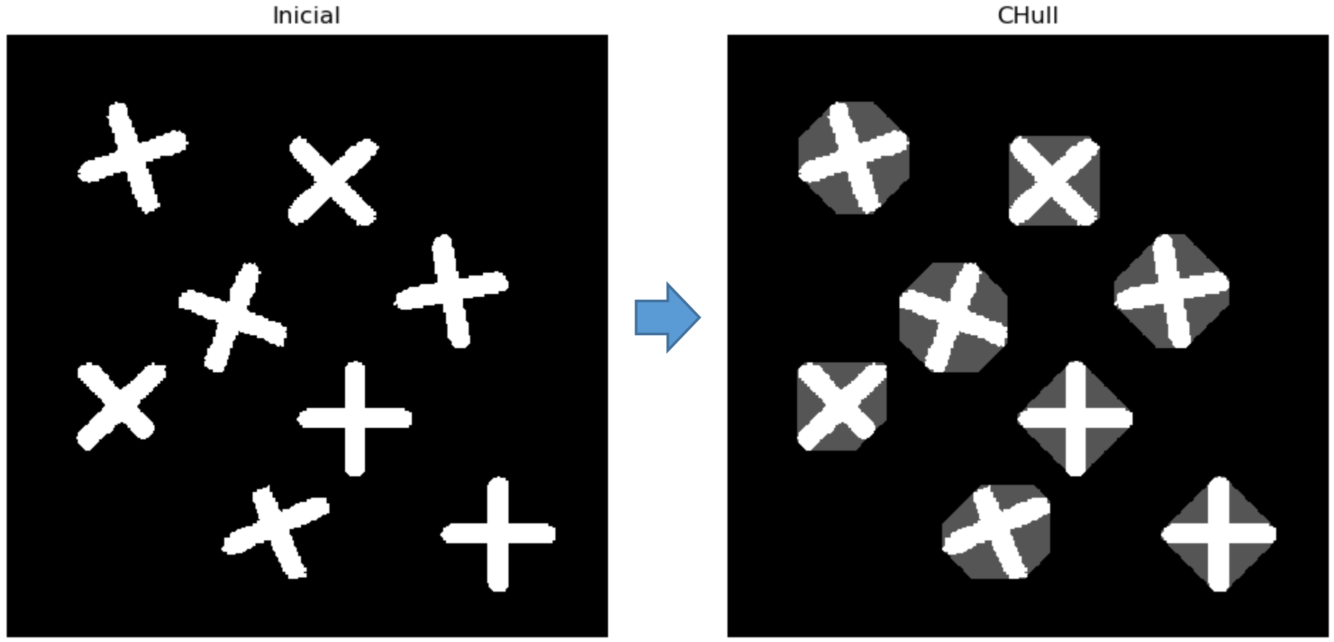
If there is a coincidence between both settings, then the pixel corresponding to the center position of B is assigned the value 1; otherwise it remains unchanged.

Morphological dilation and closure are examples of thickening transformations.

Thickening Transformation

The "**convex envelope**" (convex hull) is determined by executing the HMT transformation to determine concavities in the objects and consequent padding. The operation is iterative and will continue until it reaches idempotence.





Convex Hull

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Thickening Transformation

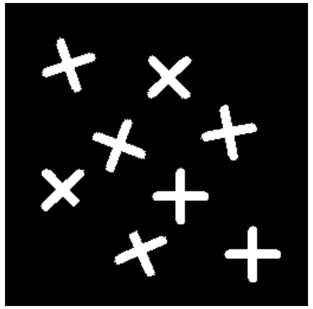
The “**skeleton by influence zones**” (SKIZ) is a skeleton structure that divides an image into regions, each containing a distinct object from the image.

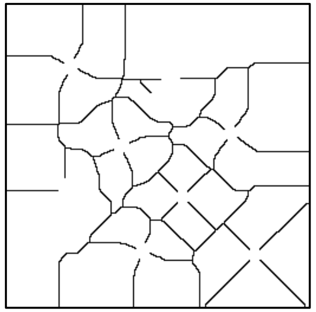
- Boundaries are defined so that all points within each area are closest to the corresponding object within that area.

- It is sometimes called the Voronoi Diagram. The operation is iterative and will continue until it reaches idempotency.

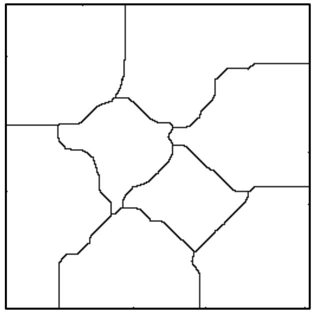
Thickening Transformation

SKIZ can be obtained by a metric process, calculating Euclidean distances, or by morphological processes, involving dilations with structuring elements of different sizes.





Background skeleton



Prunning skeleton

Binary geodetic transformations

Binary **geodetic transformations** are morphological transformations about a binary image Y, conditioned by a given binary geodesy X.

These transformations include:

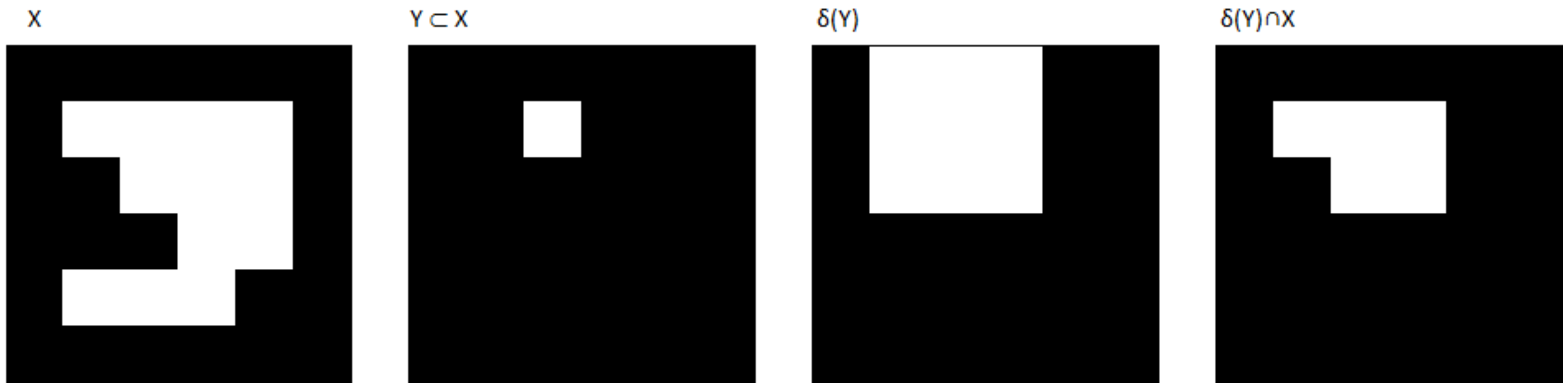
1 - Binary geodetic reconstruction by successive geodetic dilatations.

2 - Binary geodetic reconstruction by successive geodetic erosions.

Binary geodetic transformations

Geodetic dilatation: morphological dilatation of a set Y conditioned to geodesy X.

$$\boxed{\delta_X(Y) = \delta(Y) \cap X}$$



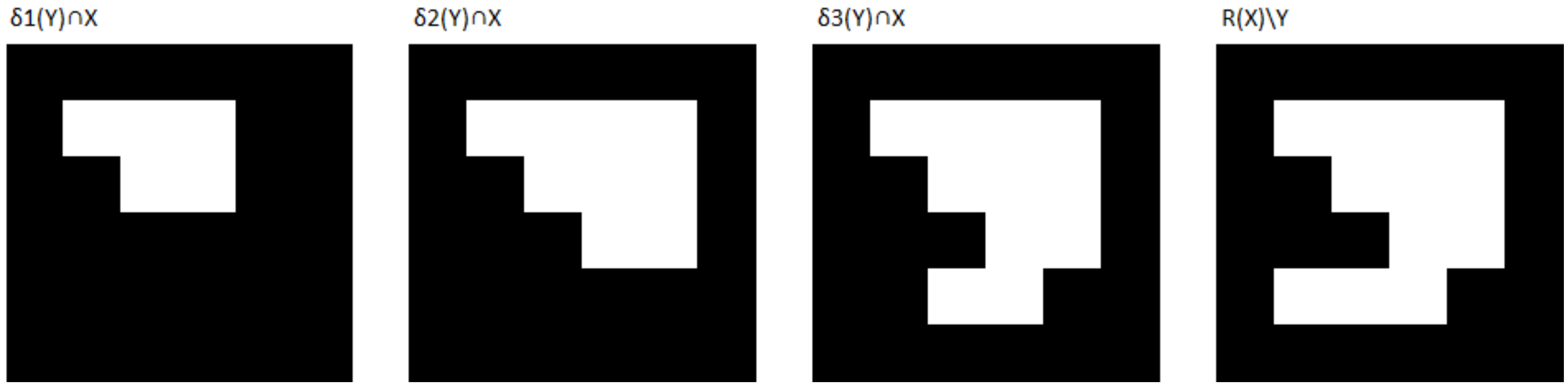
Geodesy

Marker

Binary geodetic transformations

Binary **geodetic reconstruction** by successive geodetic dilatations:

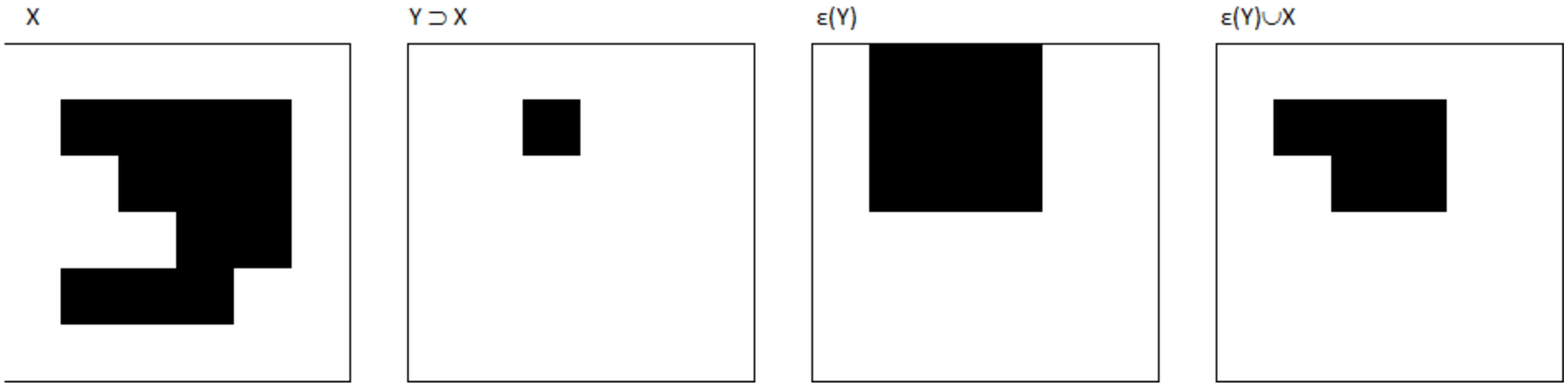
$$R_X(Y) = \delta_X^{\infty}(Y) = \lim_{n\to\infty} (\delta_X \ o \ \dots o \ \delta_X)(Y) = \delta_X^n\left(\delta_X^{n-1}\left(\delta_X^{n-2}\left(\dots\left(\delta_X^1(Y)\right)\right)\right)\right)$$



Binary geodetic transformations

Geodetic erosion: morphological erosion of a set Y conditioned to geodesy X.

$$\boxed{\varepsilon_X(Y) = \varepsilon(Y) \cup X}$$

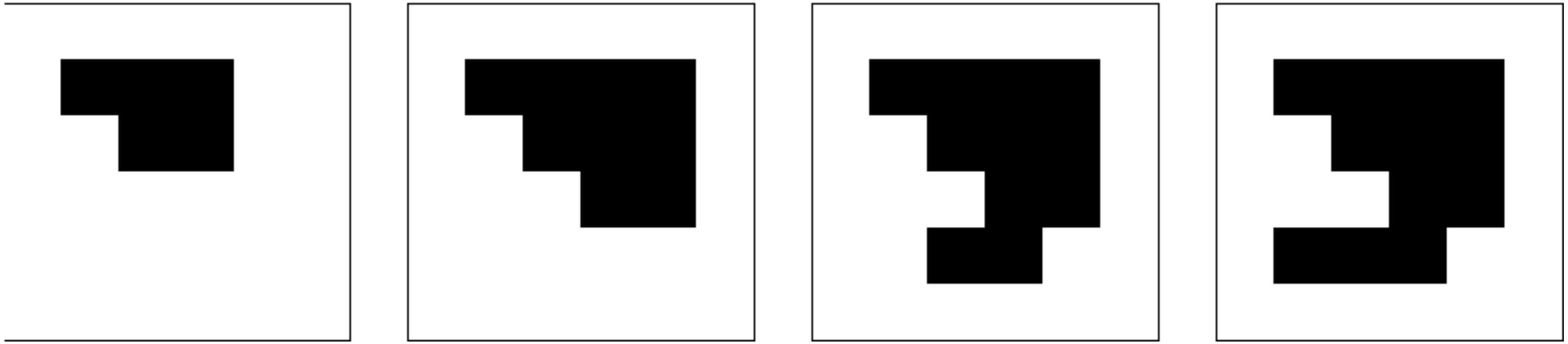


Binary geodetic transformations

Binary **geodetic reconstruction** by successive geodetic erosions:

$$R_X(Y) = \varepsilon_X^{\infty}(Y) = \lim_{n\to\infty}(\varepsilon_X \, o \, \ldots \, o \, \varepsilon_X)(Y) = \varepsilon_X^{n}\left(\varepsilon_X^{n-1}\left(\varepsilon_X^{n-2}\left(\ldots\left(\varepsilon_X^{1}(Y)\right)\right)\right)\right)$$

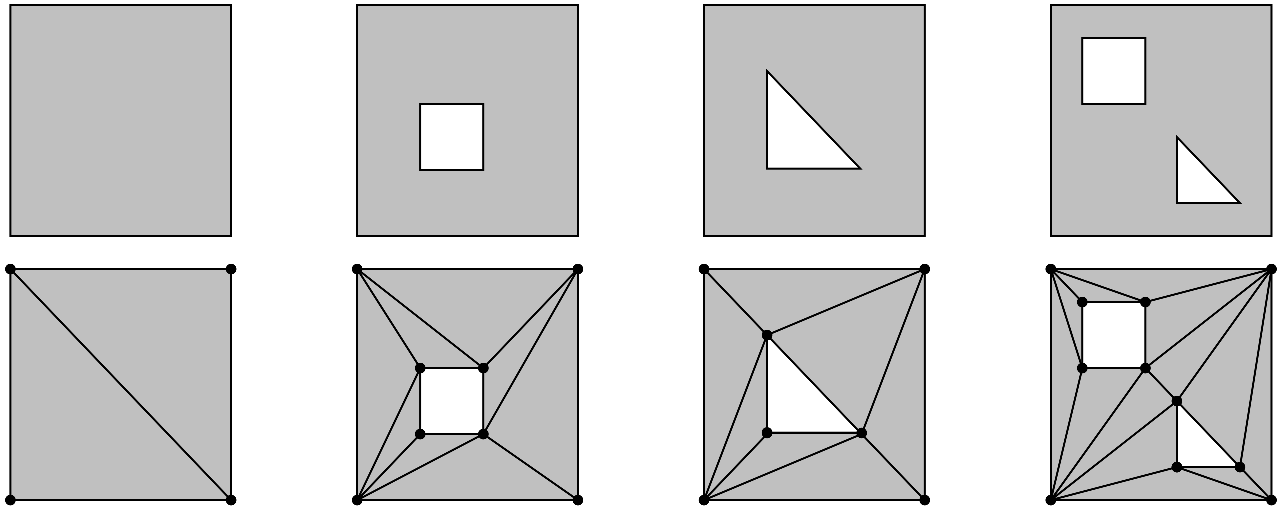
ε1(Y)∪X ε2(Y)∪X ε3(Y)∪X R(X)\Y



Connectivity Number (Euler number)

In general, the **connectivity number** of a surface or set, or also called Euler number (E), is equal to the number of vertices *v*, minus the number of edges *a*, plus the number of polygons *p*, when divided the surface. on flat polygons defined by edges and vertices.

$$E = v - a + p$$



$E = 4 - 5 + 2 = 1$ $E = 8 - 16 + 8 = 0$ $E = 7 - 14 + 7 = 0$ $E = 11 - 25 + 13 = -1$

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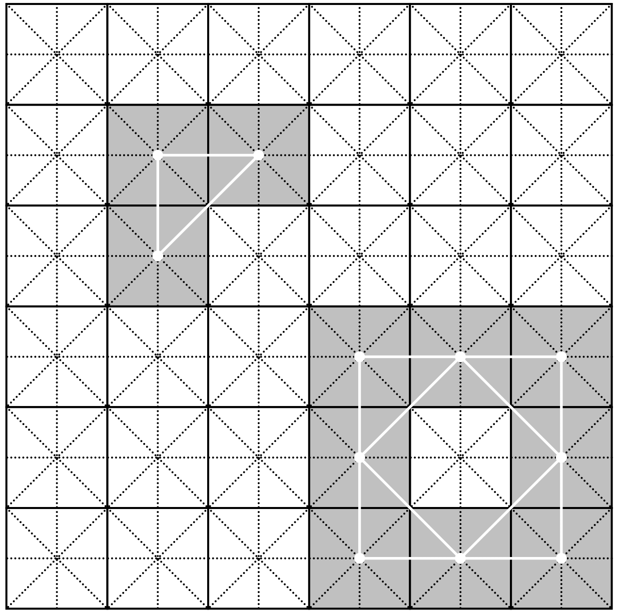
Connectivity Number (Euler number)

In the representation domain of a digital image, the value of E is determined from the pixel representation graph. The following illustrates the determination of E for the shadowed binary object.



Number of pixels = 11
Number of edges of the graph = 15
Number of triangles = 5
E = 11- 15 + 5 = 1

Number of objects = 2
Number of holes = 1

E = 2-1 = 1

$$\boxed{E = v - a + p}$$